Egy. J. Pure & Appl. Sci. 2023; 61 (3):1-10

Egyptian Journal of Pure and Applied Science

The effects of an artificial radial acceleration on the positions of L_4 and L_5 and their stability in the restricted three-body Problem

Ahmed Mostafa Kamal

Ain Shams University, Faculty of Science, Department of Mathematics, Cairo, 11517, Egypt

ARTICLE INFO

Received 8 June 2023
Accepted 12 August 2023

Keywords

Three body problem,
Lagrangian Libration points,
Artificial equilibrium points,
Stability of Libration Points.

Correspondence

Ahmed Mostafa

E-mail

ahmedmostafa@sci.asu.edu.eg

ABSTRACT

In this work, the triangular Lagrangian points L_4 and L_5 in the circular restricted three body problem are studied under the influence of applying a radial force. The equations of the *xy* planar motion of the third body with negligible mass compared to the two primaries are formulated with the addition of an external force $\alpha\boldsymbol{r}$ and solved for the case of the triangular Lagrangian points L_4 and L_5. The positions of L_4 and L_5 and the stability conditions are found. The results show the dependence of the positions of the artificial libration points on the applied external force $\alpha\boldsymbol{r}$. Also the original position of the points is shown to be a limiting case when $\alpha = 0$. The results also show the dependence of the stability conditions on α, and thus giving the possibility of discovering artificial stable libration points other than the well-known natural points. The cases of the Earth-Moon system and Sun-Jupiter system are studied numerically.

1. Introduction

The study of the three-body problem started in 1772 by Euler and Lagrange, and continued by Hill [1] followed by Poincare [2] and then many great articles in the history of celestial mechanics had been made such as Birkhoff [3], Szebehely [4], Levi-Civita [5]. When the mass of one of the three bodies is negligible with respect to the other two bodies (the primaries in this case), the problem is called the restricted three body problem (RTBP). Many important articles have been published in the RTBP, e.g. Eckstien et al. [6], Kogan [7], and Papadakis [8]. Among the most important parts of the study of the RTBP is the case of libration or equilibrium points. Those are the positions in which the body under study is not affected by the two primaries.

These points have great advantages more than any other points in the orbit of the smaller body (usually a spacecraft or a satellite). Studying stability of these points are thus very important to ensure stable equilibrium positions in the orbit that can be used for several purposes. So far five natural artificial points have been discovered, those are the three collinear points L_1, L_2 and L_3, and the two equilateral points L_4 and L_5. Concerning the study of the stability of the libration points in the RTBP, the literature is rich with many important works such as Celletti and Giorgilli [9], Leontovich [10], Arnol'd [11], and Deprit and Bartholome´ [12] who proved the stability of the equilateral triangular points in the range $0 < \mu < 0.0385208$, before Markeev [13], who excluded the two values $\mu_1 = 0.0242938$ and $\mu_2 = 0.0135160$.

The existence and stability of artificial equilibrium points in the planar case of the restricted three body problem, other than the well-known five equilibrium points, have been studied by many authors e.g. [14 – 17], also El-Saftawy et.al. [18] and Mostafa et. al. [19] used Lorentz Force to generate artificial triangular points in the planar RTBP for a charged spacecraft. In this article we study the effect of adding a radial artificial force on the position and stability of the triangular libration points in the restricted three body problem with an application to the Earth-Moon system and the Sun-Jupiter system.

2. Formulation of the problem

Consider the problem of the circular restricted three-bodies with the masses of the primaries m_1, m_2 and the mass of the small body is m in which m is negligibly small compared to m_1 and m_2. Let the primaries revolve in circular orbits around their center of mass with mean motion n and constant separation *l*. Then, $n^2 l^3 = k^2 M$ where k is the Gaussian constant and $M = (m_1 + m_2)$. Let the coordinates origin be the center of mass C.M. of m_1 and m_2, the *x* axis be the line joining m_1 and m_2, and the *y* axis be the normal to *x* in the plane of motion of the two primaries., while the *z*-axis is perpendicular to the *xy* plane. Note that if the distance *l* between m_1 and m_2 is chosen to be the unit of distance, C.M. will be located at distances m_2 and m_1 from m_1 and m_2 respectively. Fig. 1 illustrates the kinematics of the problem.

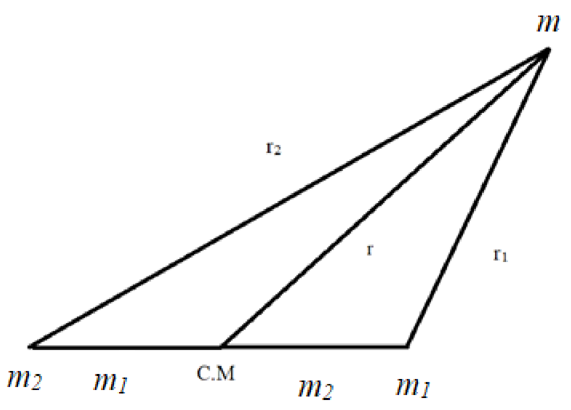

Fig. 1 The kinematics of circular planer three body.

where, **r1**= (x-m_2 , y) **r2**= (x+m_1 , y) are the relative position vectors of m with respect to m_1 and m_2 respectively, while **r** = (x, y) is the relative position vectors or m with respect to the center of mass of m_1 and m_2.

The motion of the small body is then governed in the *xy* plane by the equations:

$$\ddot{x} - 2n\dot{y} = n^2 x - k^2 \left[m_1 \frac{x-m_2}{r_1^3} + m_2 \frac{x+m_1}{r_2^3}\right], \qquad (1.1)$$

$$\ddot{y} + 2n\dot{x} = n^2 y - k^2 y \left(m_1 \frac{1}{r_1^3} + m_2 \frac{1}{r_2^3}\right), \qquad (1.2)$$

Taking m_2 = μ, m_1=1- μ, so that M =1, $n = 1$ and k=1, equations (1.1) and (1.2) will be:

$$\ddot{x} - 2\dot{y} = x - \left[(1-\mu)\frac{x-\mu}{r_1^3} + \mu \frac{x-\mu+1}{r_2^3}\right] \qquad (2.1)$$

$$\ddot{y} + 2\dot{x} = y - y\left(\frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3}\right), \qquad (2.2)$$

3. Controlling equations

By adding a radial acceleration $\alpha \boldsymbol{r}$, to the equations of motion where α is a constant, we get:

$$\ddot{x} - 2\dot{y} = x - \left[(1-\mu)\frac{x-\mu}{r_1^3} + \mu \frac{x-\mu+1}{r_2^3}\right] + \alpha\, x, \qquad (3.1)$$

$$\ddot{y} + 2\dot{x} = y - \left[(1-\mu)\frac{y}{r_1^3} + \mu \frac{y}{r_2^3}\right] + \alpha\, y, \qquad (3.2)$$

Equations (3) are the controlling equations of the motion of a small body under the effect of two primaries, with a radial force per unit mass $\boldsymbol{F} = \alpha \boldsymbol{r}$ applied to the system. The equations that locate the libration points can be written as:

$$(1+\alpha)x - \left[(1-\mu)\frac{x-\mu}{r_1^3} + \mu \frac{x-\mu+1}{r_2^3}\right] = 0 \qquad (4.1)$$

$$(1+\alpha)y - y\left[\frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3}\right] = 0 \qquad (4.2)$$

4. Artificial equilateral triangular points

This case is obtained from solving Equation (4-2) for $y \neq 0$, and setting $r_1 = r_2 = L$ hence we get,

$$1 + \alpha - \left[\frac{1-\mu}{L^3} + \frac{\mu}{L^3}\right] = 0 \qquad (5)$$

which gives $$L = (1+\alpha)^{-1/3} \qquad (6)$$

Note that, when substituting by (6) in Equation (4.1) no contradiction nor new points will arise. The position of the points L_4 and L_5 and their y-ordinates can be shown in fig. 2.

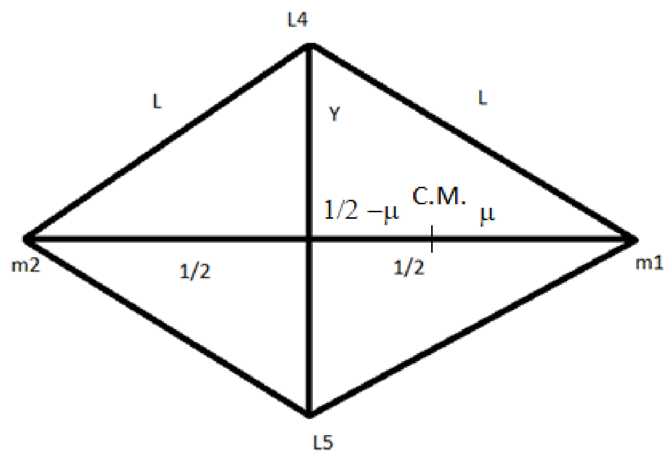

Fig. 2 Artificial L4 and L5 Points

The position of y-ordinates of L_4 and L_5 is simply $\pm\sqrt{L^2-\frac{1}{4}}$. In the ordinary case when no artificial force is applied $\alpha = 0$ and thus *L=1*, and the values of *y* for L_4 and L_5 are $\pm\frac{\sqrt{3}}{2}$. When radial acceleration $\alpha\boldsymbol{r}$ is applied, Equations (5) and (6) show the possibility of existence of new positions for the equilateral equilibrium points. The x-ordinates of all points will still be in the middle distance between the primaries m_1 and m_2, but y-ordinates will have different values according to the value of α. This is found from Fig. 3 and Equation (7)

$$y = \pm\sqrt{(1+\alpha)^{-2/3} - 1/4} \tag{7}$$

Fig. 3 shows the different possible y-ordinates for L_4 and L_5 when an external radial acceleration $\alpha\boldsymbol{r}$ is applied. As clear from Equation (7), the possible range for α is:

$$-1 < \alpha \le 7. \tag{8}$$

5. Stability of artificial triangular libration points

Equations (3.1) and (3.2) can be written as

$$\ddot{x} - 2\dot{y} = \frac{\partial U}{\partial x} \tag{9.1}$$

$$\ddot{y} + 2\dot{x} = \frac{\partial U}{\partial y} \tag{9.2}$$

with,

$$U = \frac{1}{2}(1+\alpha)(x^2+y^2) + \frac{(1-\mu)}{r_1} + \frac{\mu}{r_2}. \tag{9.3}$$

Linearizing Equations (9.1) and (9.2) about the equilibrium points (x_0, y_0) by:

$$x = x_0 + \varepsilon\xi, \quad y = y_0 + \varepsilon\eta, \qquad \varepsilon \ll 1. \tag{10}$$

where $\varepsilon\xi$ and $\varepsilon\eta$ are the small displacements of m from the libration point (x_0, y_0) along the x-direction and y-direction respectively.

Then, by expanding the equations keeping only first order terms, we get:

$$\ddot{\xi} - 2\dot{\eta} = a\,\xi + b\,\eta, \tag{11.1}$$

$$\ddot{\eta} + 2\dot{\xi} = b\,\xi + c\,\eta, \tag{11.2}$$

The coefficients a, b and c are given by,

$$a = \left.\frac{\partial^2\phi}{\partial x^2}\right|_{(x_0,y_0)}, b = \left.\frac{\partial^2\phi}{\partial x\partial y}\right|_{(x_0,y_0)}, c = \left.\frac{\partial^2\phi}{\partial y^2}\right|_{(x_0,y_0)} \tag{11.3}$$

Then the characteristic equation for the system (11.1 – 11.2) is,

$$\lambda^4 + \lambda^2(4-a-c) + a\,c - b^2 = 0. \tag{12}$$

which can be written as:

$$\lambda^4 + F\lambda^2 + H = 0, \tag{13.1}$$

with,

$$F = (4-a-c), \tag{13.2}$$

$$H = a\,c - b^2. \tag{13.3}$$

The condition for λ to be imaginary is that λ^2 is a real negative value. This is guaranteed by the satisfaction of the conditions [18]:

$$F > 0, \tag{14.1}$$

$$H > 0. \tag{14.2}$$

$$F^2 - 4H > 0, \tag{14.3}$$

Inequalties (14.1), (14.2) and (14.3) are the necessary conditions for the artificial libration points to be stable.

If the distance from the original triangular libration points to the small primary is *L* = 1, the *x* and *y* coordinates $x_{4,5} = -(0.5-\mu)$ and $y_{4,5} = \pm\frac{\sqrt{3}}{2}$. These values will be changed to $L = (1+\alpha)^{-1/3}$ and $y_{4,5} = \pm\sqrt{(1+\alpha)^{-2/3} - 1/4}$ due to the existence of the radial acceleration αr. Using the values ($\mu - 0.5, \pm\sqrt{(1+\alpha)^{-2/3} - 1/4}$) for (x_0, y_0) in Equation (11.3), we get the values of *F* and *H* (after simplification) as functions of α and μ in Equations (15.1) and (15.2),

$$F = 1 - 3\alpha \tag{15.1}$$

$$H = \frac{9}{4}\mu(1-\mu)\left(4-(1+\alpha)^{2/3}\right)(1+\alpha)^{8/3} \tag{15.2}$$

Fig. 4, 5 and 6 show the graphs of *F*, *H*, and $F^2 - 4H$, respectively with α and μ.

Fig. 4 shows the linear dependence of *F* on α. Two important cases to be considered which are the Earth-Moon system ($\mu = 0.012$) and Jupiter – Sun system ($\mu = 0.00095$). Equation (15.1) and Fig. 4 show that no difference will exist for *F* between the two systems or any other system due to the independence of *F* on μ. Fig. (5) and (6) show the nonlinear dependence of *H* and $F^2 - 4H$ on both α and μ.

From Equation (15.1), the condition for *F* > 0 is $\alpha < \frac{1}{3}$.

Then from Equation (8), the range of α is reduced to:

$$-1 < \alpha < \frac{1}{3} \tag{16}$$

The analysis for *H* shows that it is always positive in the above range regardless the value of , noting that $0 < \mu < 1$ and that for the range in (16): $1 + \alpha > 0$ and $(1+\alpha)^{2/3} < 4$. Fig. 7 and Fig. 8 show the range of positive values for *F* and *H*. It remains to find the range of positive values of $F^2 - 4H$. Let's denote it *G* to start the analysis. From Equation (15.1) and Equation (15.2),

$$G = (1-3\alpha)^2 - 9\mu(1-\mu)(4-(1+\alpha)^{2/3})(1+\alpha)^{8/3} \tag{17}$$

For G to be greater than zero,

$$\frac{(1-3\alpha)^2}{9(4-(1+\alpha)^{2/3})(1+\alpha)^{8/3}} - \mu(1-\mu) > 0$$

$$\mu^2 - \mu + \frac{(1-3\alpha)^2}{9(4-(1+\alpha)^{2/3})(1+\alpha)^{8/3}} > 0$$

$$\mu^2 - \mu + f(\alpha) > 0 \qquad (18.1)$$

$$f(\alpha) = \frac{(1-3\alpha)^2}{9(4-(1+\alpha)^{2/3})(1+\alpha)^{8/3}} \qquad (18.2)$$

This can be factorized to (μ-f_1) (μ-f_2) > 0 which will be satisfied in the two regions μ < Min (f_1 , f_2) and μ > Max(f_1,f_2), where f_1 and f_2 satisfy the two conditions

$$f_1 + f_2 = 1, \qquad (19.1)$$

$$f_1 f_2 = f(\alpha), \qquad (19.2)$$

keeping in mind that -1 < α < 1/3. Solving Equation (19.1) and Equation (19.2), we get

$$f_1 = \frac{1}{2}\left(1 - \sqrt{1 - \frac{4(1-3\alpha)^2}{9(1+\alpha)^{\frac{8}{3}}(4-(1+\alpha)^{\frac{3}{2}})}}\right) \qquad (20.1)$$

$$f_2 = \frac{1}{2}\left(1 + \sqrt{1 - \frac{4(1-3\alpha)^2}{9(1+\alpha)^{\frac{8}{3}}(4-(1+\alpha)^{\frac{3}{2}})}}\right) \qquad (20.2)$$

Since f_1< f_2, the regions for stability are μ < f_1 and μ > f_2. Also, f_1 and f_2 are real for positive or zero values only of under the root. Solving numerically, we find that the value under the root is non-negative in the range:

$$-0.2703 < \alpha < 1.34 \qquad (21)$$

Thus, from Inequalities (16) and (21), final range of α for possible artificial equilateral triangular points is:

$$-0.2703 < \alpha < 0.3333 \qquad (22)$$

Fig. 9 and Fig. 10 illustrate the stability region in the $\alpha - \mu$ plane. It is to be noted that the solutions for f_1 and f_2 give the regions of stable solutions for μ and $1-\mu$. Fig. 9 shows the region of possible stable triangular points in the $\alpha - \mu$ plane in the sense that each value of α is valid for the corresponding value of μ and all the values of μ less than it, in other words every point on the curve gives the value of α with the maximum value of μ that can have stable triangular Lagrangian points for such α-value. Also, for every value of μ, the corresponding value of α and all the values to the left on α-axis (bigger numeric values in the negative direction) will generate stable triangular points.

This trend is valid till the valueμ=0.0385 and α=0, the case of natural stability [3], [12]. For the values of μ less than this value, stability is still guaranteed for all values of α under the curve and the values to the left of it including the zero value, thus giving the natural case as a limiting case of this model by setting α=0.

6. Numerical investigations for the cases of the Earth-Moon and Sun-Jupiter systems

For the earth-moon system the parameter μ =0.012, is substituted in Equations (15.1), (15.2) and Equation (17). As we studied before, F and H are not affected, but for G, we search numerically for its positive values in the range [-0.2703, 3333] of α, to get α<0.1174. This gives the range for stable triangular points in the case of Earth-Moon system:

$$-0.2703<\alpha<0.1174 \qquad (23)$$

The above range for α coincides with Fig. 9 at μ =0.012. Figure 11 shows artificial triangular stable Lagrangian points for the earth-moon system with different values of α. Doing the same process as above but with μ =0.00095 to study the case of Sun-Jupiter, we get α<0.2626, thus giving the range for stable triangular points in the case of Sun-Jupiter system:

$$-0.2703<\alpha<0.2626$$

Fig. 12 shows artificial triangular stable Lagrangian points for the earth-moon system with different values of α.

7. Conclusion

In this article, we studied the possibility of generating artificial triangular Libration points L4 and L5 in the circular restricted three body problem, by introducing an artificial radial acceleration αr to the model, where r is the position vector measured from the center of mass of the two primaries, and α is a parameter whose value is studied for guaranteeing the generation of L4 and L5 as well as their stability. The results show the possibility of generating artificial L4 and L5 points for the range -1<α≤7. Part of this range contains stable points L4 and L5 namely -0.27<α≤0.333 depending on the value of the mass ratio μ. In the classical problem stable L4 and L5 points can exist for μ≤0.0385, but with the existence of the radial acceleration αr stable L4 and L5 points can exist for other different values of μ. The results show the consistency of this model with the classical case when α=0. The problem has been studied in the two-dimensional case, but it can be extended to the three- dimensional case for a future study. Also, the work can be extended to include other affecting forces besides or in place of the force in the radial direction.

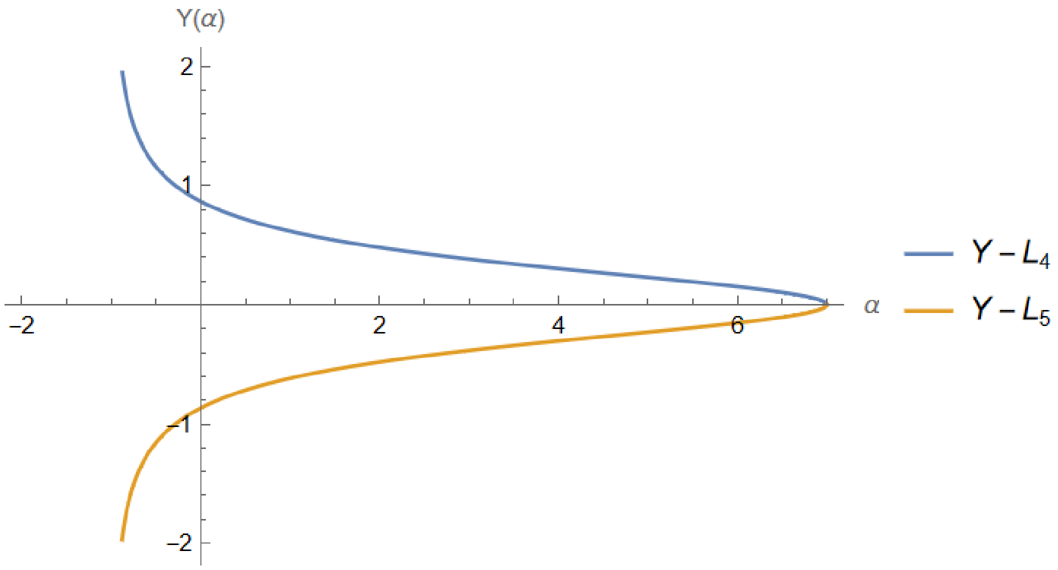

Fig. 3 Y-ordinates of artificial L_4 and L_5

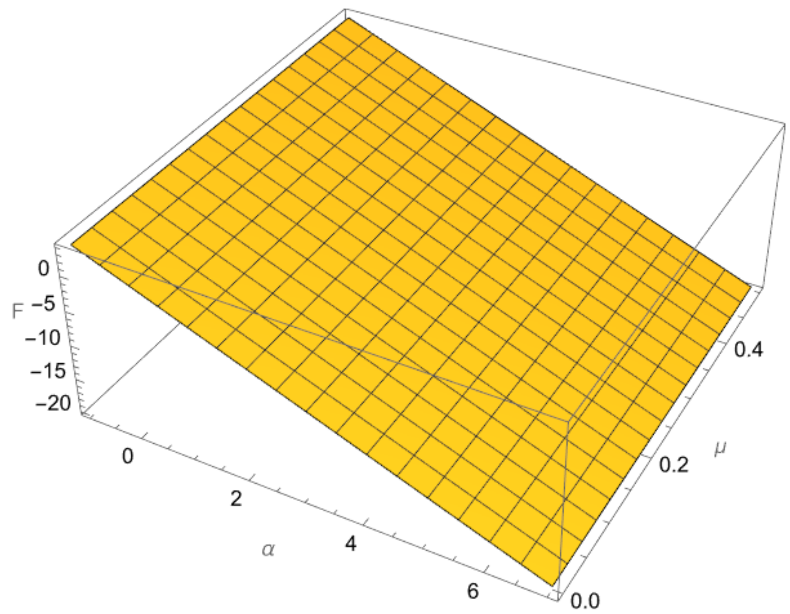

Fig. 4 Relation between the stability parameter F and the force constant α

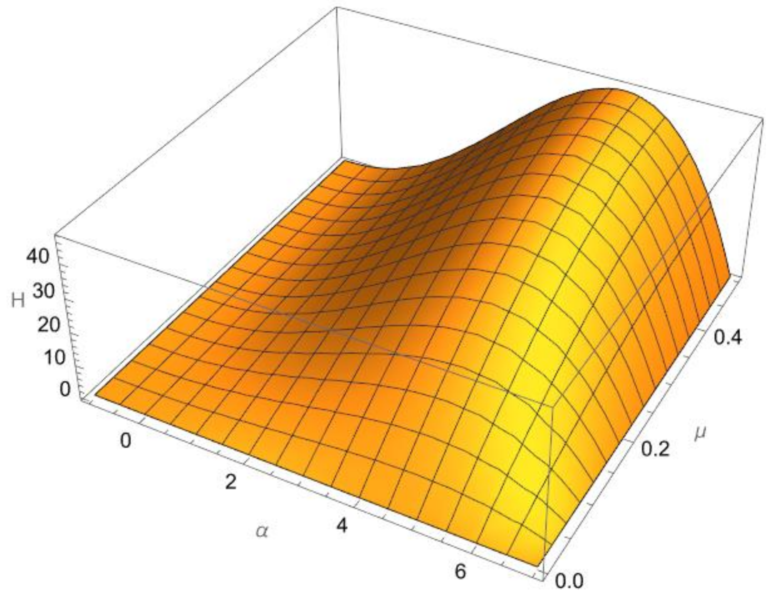

Fig. 5 Relation between the stability parameter H with the force constant α and the mass ratio μ

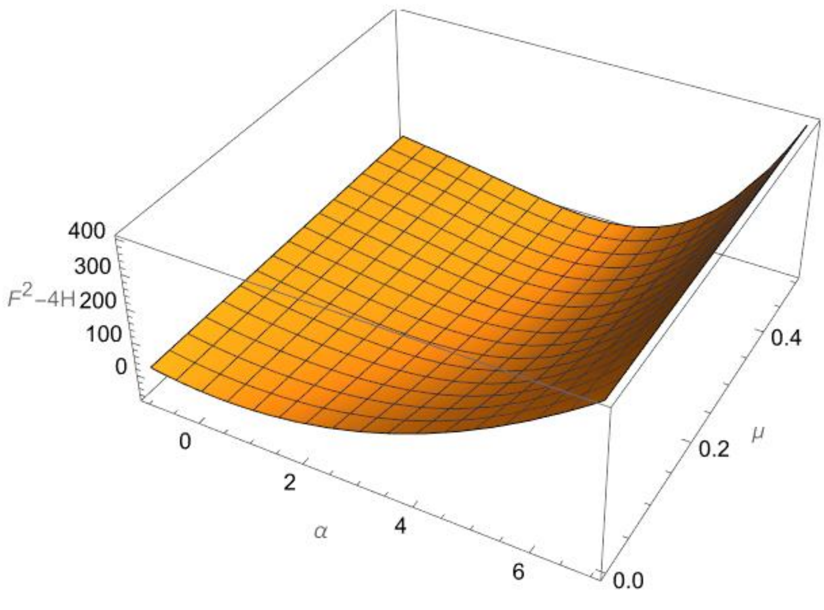

Fig. 6 Relation between $F^2 - 4H$ with the force constant α and the mass ratio μ

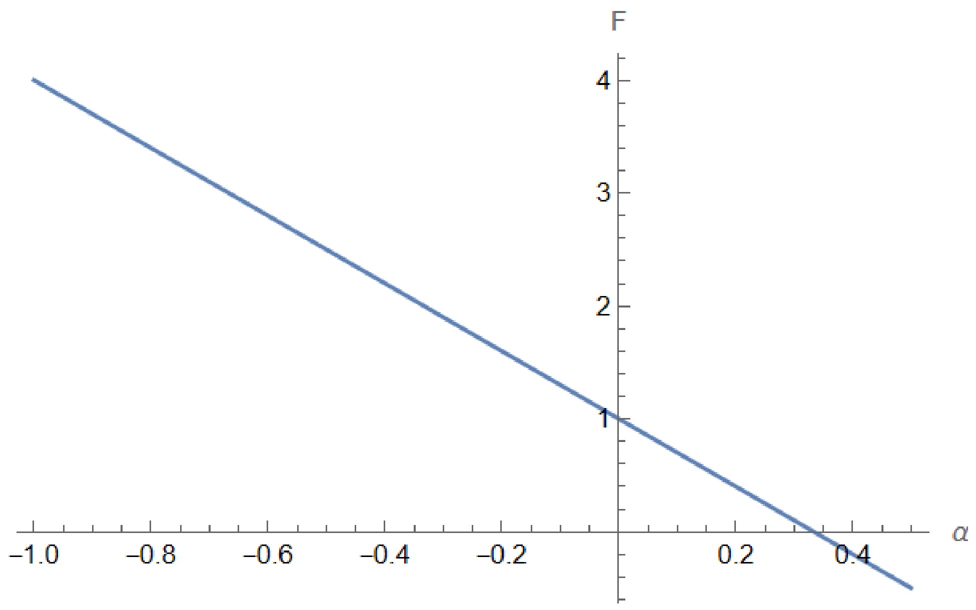

Fig. 7 The range of positive values of F

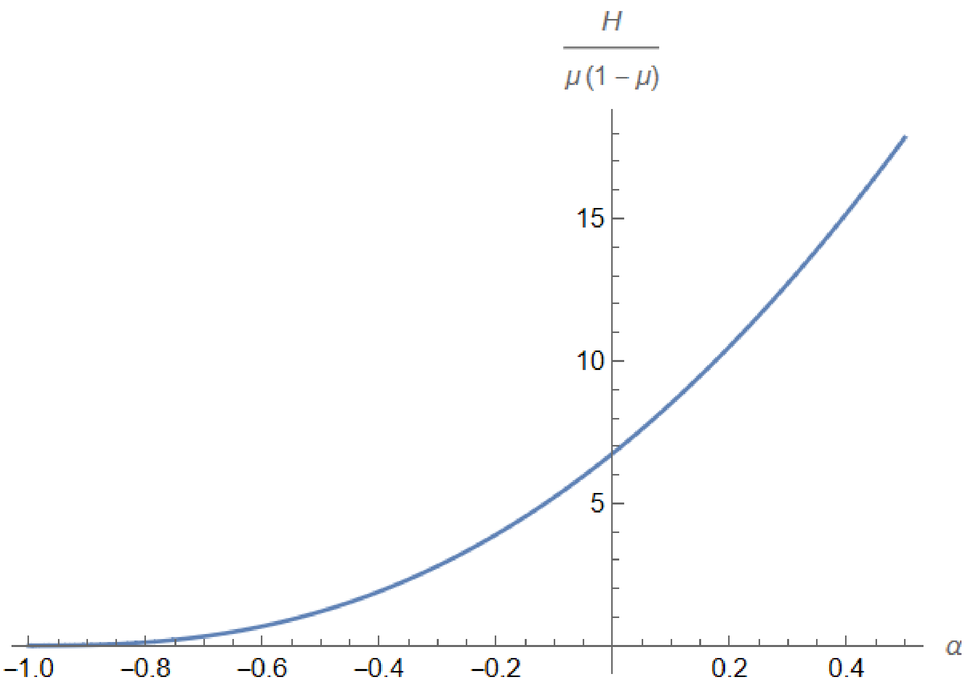

Fig. 8 The range of positive values of H

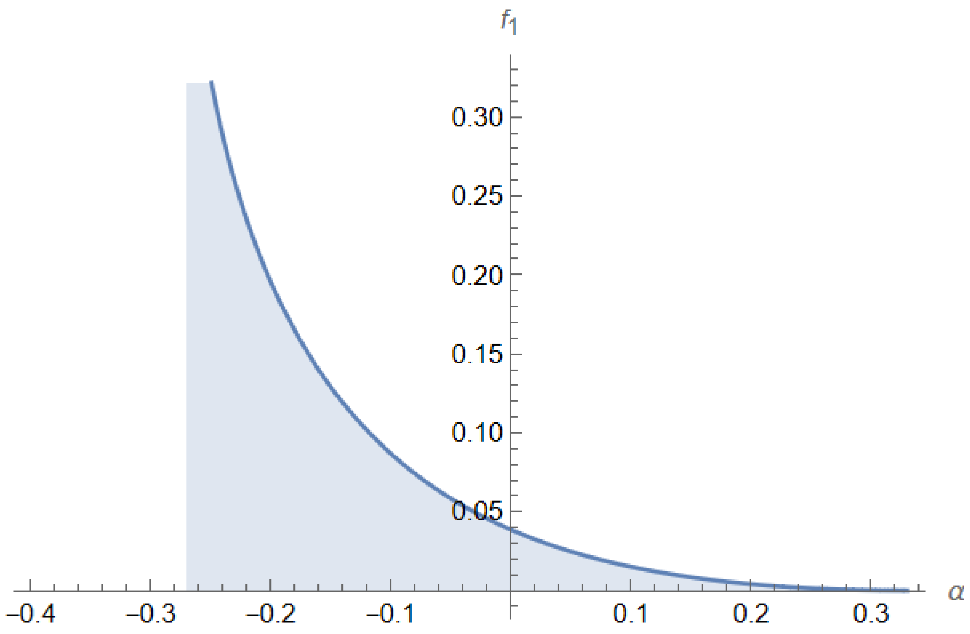

Fig. 9 The stability region in $\alpha - \mu$ plane ($\mu < 1/2$)

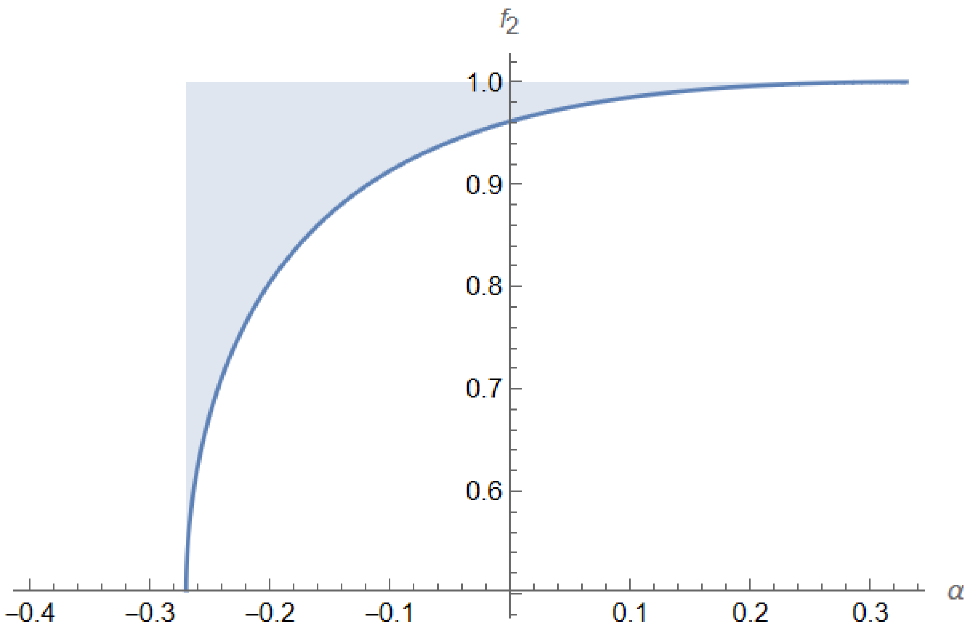

Fig. 10 The stability region in $\alpha - \mu$ plane ($\mu > 1/2$)

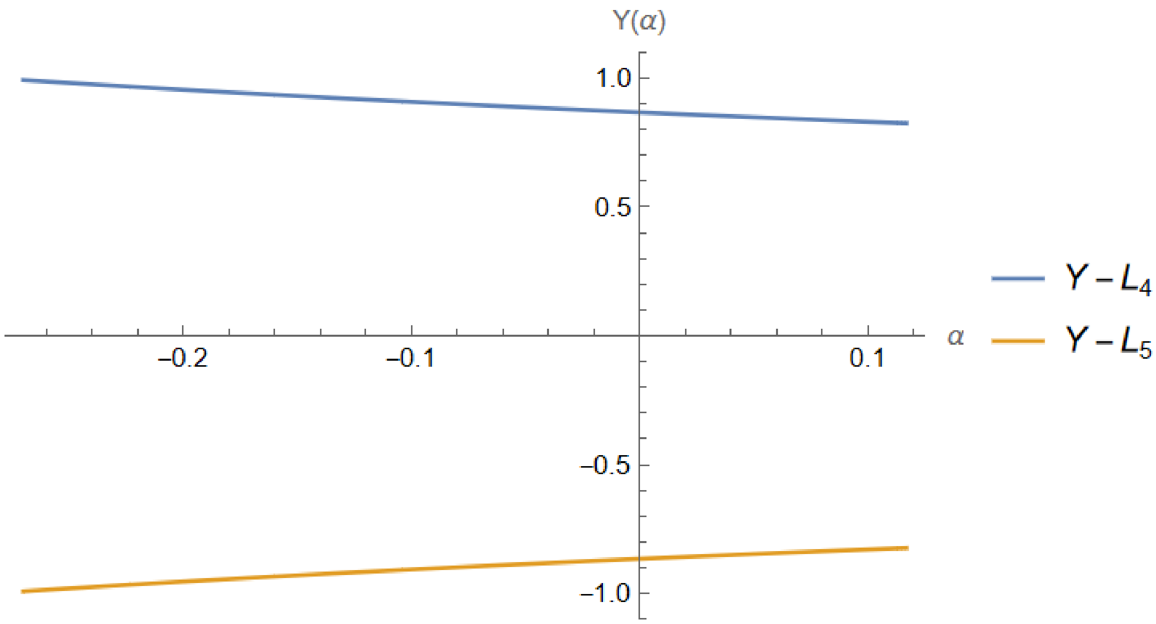

Fig. 11 Stable Triangular Lagrangian points for the Earth-Moon system

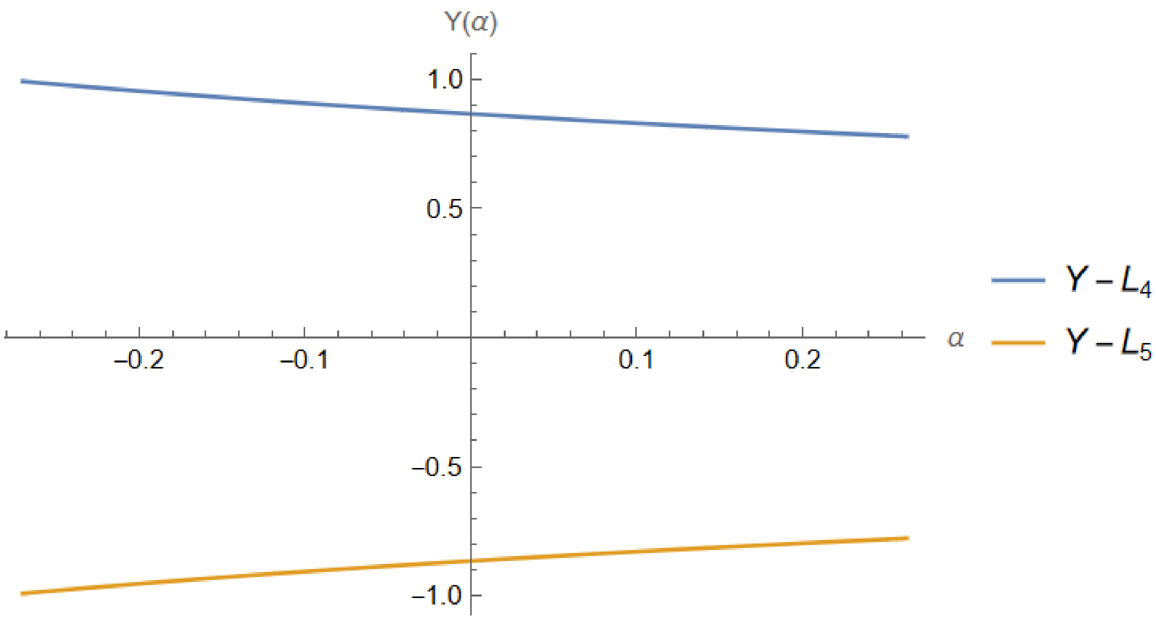

Fig. 12 Stable Triangular Lagrangian points for the Sun-Jupiter system

8. Reference

1. **G. W., Hill. (1878).** Researches in the Lunar Theory, American Journal of Mathematics, **Vol. 1, No. 1, 5-26. doi:10.2307/2369430**
2. **Poincaré, H. (1899).** Les Methodes Nouvelles de la Mécanique. Also reprinted by Springer, Berlin in **2007**.
3. **G. D., Birkhoff. (1927).** Dynamical Systems, American Mathematical Society, Providence, RI, USA.
4. **Szebhley, V. (1967).** Theory of orbits:The Restricted Problem of Three Bodies, Academic press.
5. **T., Levi-Civita. (1920).** Sur la r´egularisation du probl`eme des troiscorps, Acta Mathematica, **vol. 42, 99–144.**
6. **Eckstein, M. C., Shi, Y. Y., Kevorkian, J.(1966).** A Uniformly Valid Asymptotic Representation of Satellite Motion Around the Smaller Primary in the Restricted Three-Body Problem, Progress in Astronautics and Rocketry,**Vol 17, 183-198.**
7. **Kogan, A. Yu. (1989).** Distant satellite orbits in the restricted circular three-body problem, Cosmic Res., **Vol. 26, No. 6, 705 – 710**
8. **K. E., Papadakis. (2004).** The 3D restricted three-body problem under angular velocity variation, Astronomy and Astrophysics, **Vol. 425 , No. 3 , 1133 - 1142**
9. **A., Celletti., A., Giorgilli. (1990).** On the stability of the Lagrangian points in the spatial restricted problem of three bodies, Celestial Mechanics and Dynamical Astronomy **vol. 50, 31–58**
10. **A. M., Leontovich. (1962).** On the stability of the Lagrange periodic solution for the reduced problem of three bodies, Dokl. Akad . Nauk SSSR, **V. 143, N. 3, 525–528.**
11. **V. I., Arnol'd. (1961).** The stability of the Fuilibrium position of a Hamiltonian system of ordinary differential equations in the general elliptic case, Dokl. Akad. Nauk SSSR, **V. 137, N. 2, 255–257.**
12. **Deprit., A. & Deprit-Bartholomé., A. (1967).** Stability of the triangular Lagrangian points, Astron. J. **72, 173–179.**
13. **A. P., Markeev. (1969).** On the stability of the triangular libration points in the circular bounded three-body problem, Prikl. Matern. Mekhan, **vol. 33, 112 - 116.**
14. **A. A., Perezhogin. (1976).** Stability of the sixth and seventh libration points in the photo-gravitational restricted circular three-body problem, Soviet Astronomy Letters, **vol. 2, pp. 174-175.**
15. **M. Y., Morimoto., H., Yamakawa., & K., Uesugi. (2007).** Artificial equilibrium points in the low-thrust restricted three-body problem, Journal of Guidance, Control, and Dynamics, **vol. 30, no. 5, 1563–1568.**
16. **J., Bookless., & C., McInnis. (2008).** Control of lagrange point orbits using solar sail propulsion, Acta Astronautica, **vol. 62, no. 2, 159–176.**
17. **S., Baig. & C. R., McInnes. (2008).** Artificial three-body equilibria for hybrid low-thrust propulsion, Journal of Guidance, Control, and Dynamics, **vol. 31, no. 6, 1644–1655.**
18. **M. I., El-Saftawy., M. A., Yousef., & A., Mostafa. (2022).** Artificial Controlling of the Collinear Libration Points Using Lorentz Force in the Restricted Three-Body Problem, Advances in Astronomy,Volume **2022**, Article **ID 1445354.**
19. **A., Mostafa., M. A., Yousef. , & M. I. El-Saftawy. (2022).** Artificial Triangular Points by Lorentz Force in the Restricted Three-Body Problem, Advances in Astronomy, Volume **2022,** Article **ID 4157792.**